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$X \subset \mathbb{P}^N$: a non-dege proj variety over $k = \bar{k}$ with $p = \mathrm{char}(k) \geqslant 0$.

Def (Segre locus). $\pi_z : \mathbb{P}^N \setminus \{z\} \to \mathbb{P}^{N-1}$: the proj from a pt $z \in \mathbb{P}^N$.

$$\mathfrak{S}^{\mathrm{out}}(X) := \{z \in \mathbb{P}^N \setminus X \mid \boldsymbol{\pi_z|_X : X \to \pi_z(X)} \text{ is } \textbf{not} \text{ birational}\}^-,$$

$$\mathfrak{S}^{\mathrm{inn}}(X) := \{z \in X \mid \boldsymbol{\pi_z|_X : X \setminus \{z\} \to \pi_z(X)} \text{ is } \textbf{not} \text{ birational}\}^-.$$

Theorem A (B. Segre, 1936). Assume $\boldsymbol{p = 0}$. Then,
$\mathfrak{S}^{\mathrm{out}}(X) =$ (finitely many **linear subspaces** in $\mathbb{P}^N$).

- Recently, $\mathfrak{S}^{\mathrm{out}}(X)$, $\mathfrak{S}^{\mathrm{inn}}(X)$, and related topics have been studied by A. Calabri-C. Ciliberto, E. Ballico, A. Noma, ...

Q. Does the **linearity** of $\mathfrak{S}^{\mathrm{out}}(X) \cup \mathfrak{S}^{\mathrm{inn}}(X)$ hold in **char** $\boldsymbol{p \geqslant 0}$?

Theorem 1 (F). Assume $\boldsymbol{p \geqslant \deg(X)}$ or $\boldsymbol{p = 0}$. Then,
$\mathfrak{S}^{\mathrm{out}}(X) \cup \mathfrak{S}^{\mathrm{inn}}(X) =$ (finitely many **linear subspaces** in $\mathbb{P}^N$).

Example 2 ($\boldsymbol{p < \deg(X)}$, Fukasawa-F). ℓ : a prime number,
$X := (x_0^\ell - x_1 x_3^{\ell-1} = x_1^\ell - x_2 x_3^{\ell-1} = 0) \subset \mathbb{P}^3$ (a rational curve of deg $\boldsymbol{\ell^2}$).

1. Either $p \neq \ell$ or $p = 0 \Longrightarrow \mathfrak{S}^{\mathrm{out}}(X) = \{(1,0,0,0)\}$,
   Moreover, $\ell \geqslant 3 \Longrightarrow \mathfrak{S}^{\mathrm{inn}}(X) = \{(0,0,0,1)\}$.
2. $\boldsymbol{p = \ell} \Longrightarrow \mathfrak{S}^{\mathrm{out}}(X) = (x_3 = x_0^\ell x_2 - x_1^{\ell+1} = 0)$, **(non-linear)**
   Moreover, $\boldsymbol{p = \ell \geqslant 3} \Longrightarrow \mathfrak{S}^{\mathrm{inn}}(X) = X$.
3. $\ell = 2$ & $p \geqslant 0 \Longrightarrow \mathfrak{S}^{\mathrm{inn}}(X) = \emptyset$.

Rem. Linearity of $\mathfrak{S}^{\mathrm{out}}(X) \cup \mathfrak{S}^{\mathrm{inn}}(X) \Longrightarrow$ linearity of $\mathfrak{S}^{\mathrm{out}}(X)$.
($\because$ $\forall$ irred comp $Z \subset \mathfrak{S}^{\mathrm{out}}(X)$, $Z =$ (an irred comp of $\mathfrak{S}^{\mathrm{out}}(X) \cup \mathfrak{S}^{\mathrm{inn}}(X)$).)

Rem. The following is an essential fact in $\boldsymbol{p = 0}$:

For a variety $Y \subset \mathbb{P}^N$, and for a linear subspace $L \subset \mathbb{P}^N$,
($\diamond$) $\quad \dim \pi_L(Y \setminus L) = \dim(Y) - \dim(\mathbb{T}_y Y \cap L) - 1$ ($y \in Y$: general pt).

In $\boldsymbol{p > 0}$, equality ($\diamond$) does *not* hold in general, since the **separability of the map** π_L is *not* guaranteed.

Our strategy to prove Thm 1 & Ex 2.

($\star$) Assume $p \geqslant 0$ & $\mathbf{codim}(\boldsymbol{X}, \mathbb{P}^N) = \mathbf{2}$.
Give **a method to compute the defining ideal** of $\mathfrak{S}^{\mathrm{out}}(X) \cup \mathfrak{S}^{\mathrm{inn}}(X)$.

$\Downarrow$

Assume $p \geqslant \deg(X)$ & $\mathbf{codim}(\boldsymbol{X}, \mathbb{P}^N) = \mathbf{2}$.
Show the **linearity** of $\mathfrak{S}^{\mathrm{out}}(X) \cup \mathfrak{S}^{\mathrm{inn}}(X)$.

$\Downarrow$

By **induction** on $\mathbf{codim}(\boldsymbol{X}, \mathbb{P}^{\boldsymbol{n}})$, show the **linearity** of $\mathfrak{S}^{\mathrm{out}}(X) \cup \mathfrak{S}^{\mathrm{inn}}(X)$. **(Thm 1)**

$\Downarrow$

For X in Ex 2, calc the ideal of $\mathfrak{S}^{\mathrm{out}}(X) \cup \mathfrak{S}^{\mathrm{inn}}(X)$.
$\rightsquigarrow$ The **non-linearity** is shown. **(Ex 2)**

Defining ideal of $\mathfrak{S}^{\mathrm{out}}(X) \cup \mathfrak{S}^{\mathrm{inn}}(X)$.

X : not a cone & $\mathrm{codim}(X, \mathbb{P}^N) = 2$. For $e \in \mathbb{N}$,

$$\mathbf{Loc}_{\boldsymbol{e}}(\boldsymbol{X}) := \{z \in \mathbb{P}^N \mid \deg(\pi_z(X \setminus \{z\}))^- \leqslant e\}.$$

Prop 3. For every irred comp $\boldsymbol{Z}$ of $\mathfrak{S}^{\mathrm{out}}(X) \cup \mathfrak{S}^{\mathrm{inn}}(X)$, $\exists\, e < \deg(X)$ s.t. $\boldsymbol{Z} =$ (an **irred comp** of $\mathbf{Loc}_{\boldsymbol{e}}(\boldsymbol{X})$).

Thm 4. $r := h^0(\mathbb{P}^N, \mathcal{I}_X(e))$. Then,

$$\mathbf{Loc}_{\boldsymbol{e}}(\boldsymbol{X}) = \begin{pmatrix} \text{the zero set of all} \\ r \times r \text{ minors of } \boldsymbol{\Lambda(e)} \end{pmatrix},$$

where $\Lambda(e)$ is constructed as follows.

Construction of the matrix $\boldsymbol{\Lambda(e)}$.

$\mathbb{P}^N = \{(x_0, x_1, \dots, x_N)\}$,
$\boldsymbol{i} = (i_0, i_1, \dots, i_N) \in \mathbb{Z}_{\geqslant 0}^{N+1}$.
For $0 \leqslant k \leqslant N$,
$\omega_k := (0, \dots, 0, \overset{k}{\check{1}}, 0, \dots, 0)$.

For each $\boldsymbol{i}$,

$$D_{\boldsymbol{i}} := \frac{1}{i_0! i_1! \cdots i_N!} \left(\frac{\partial}{\partial x_0}\right)^{i_0} \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_N}\right)^{i_N}$$

(iterative higher derivation).

Def 5. $\{h_1, \dots, h_r\}$: a basis of $H^0(\mathbb{P}^N, \mathcal{I}_X(e))$. For an integer $s \leqslant e$,

$$\Lambda_{e-s} := \begin{bmatrix} D_{(e-s)\omega_0} h_1 & D_{(e-s-1)\omega_0 + \omega_1} h_1 & \cdots & D_{\boldsymbol{i}} h_1 & \cdots & D_{(e-s)\omega_N} h_1 \\ D_{(e-s)\omega_0} h_2 & D_{(e-s-1)\omega_0 + \omega_1} h_2 & \cdots & D_{\boldsymbol{i}} h_2 & \cdots & D_{(e-s)\omega_N} h_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_{(e-s)\omega_0} h_r & D_{(e-s-1)\omega_0 + \omega_1} h_r & \cdots & D_{\boldsymbol{i}} h_r & \cdots & D_{(e-s)\omega_N} h_r \end{bmatrix}_{(|\boldsymbol{i}| = e - s)}.$$

$$\boldsymbol{\Lambda(e)} := \begin{cases} \Lambda_{e-1} & (\boldsymbol{p} > e \text{ or } \boldsymbol{p} = 0), \\ [\Lambda_{e-1}\ \Lambda_{e-\boldsymbol{p}}\ \Lambda_{e-2\boldsymbol{p}}\ \cdots\ \Lambda_{e-\lfloor e/\boldsymbol{p} \rfloor \cdot \boldsymbol{p}}] & (\boldsymbol{p} \leqslant e). \end{cases}$$